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# EQUATIONS FOR ALLUVIAL SOIL STORAGE COEFFICIENTS

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**Abstract**

The paper defines the average and local soil storage coefficients, for a soil with relatively shallow water table. By means of an adequate law for the suction distribution on soils level and certain parametrical models for the moisture retention curve of soil, the analytical equations for these storage coefficients were determined. Theoretical results are to be applied for a case study and on the basis of a pertinent analysis, and some recommendations were made for the practical use of formulae proposed for the storage coefficients.

*Keywords:* analytical equations, mathematical model, soil, storage coefficients

## 1. Introduction

The soil storage coefficients are mainly used within the mathematical modeling of the draining of a soil profile. In text-books (Kabat, 1998) there is defined an unique storage coefficient, $\mu$, by means of the ratio between the change in soil moisture storage of the profile, $\Delta W$, [L], and the corresponding change in water table depth, $\Delta Z$, [L]. Anyway, after a more thorough analysis it can be seen that the significance and/or the value of coefficient $\mu$ are depending on the next factors: a) the original water table depth compared to the land's area, $Z^*$; b) the sense of depth change $\Delta Z$ ($\Delta Z>0$ – decrease; $\Delta Z<0$ – increase); c) the nature of change nature $\Delta Z$ (finite or infinitesimal).

In the process of assessing the $\mu$ coefficient the next simplifying assumptions are accepted: 1) the whole soil profile is located in the capillary fringe's area; 2) hydrodynamic balance (constant hydrodynamic load upon the whole soil profile) – this, in both conditions: in initial condition (water table at depth $Z^*$), and, as well, in the final condition (water table stabilized at depth $Z^*+\Delta Z$).

For a water tables finite descent $\Delta Z$, has been defined ***the soil storage coefficient***, $\overline{\mu}$, average on depth $\left[Z^*, Z^*+\Delta Z\right]$, by mean of next equation:

$$\overline{\mu} = \frac{\Delta W_{1\_2}}{\Delta Z} = \frac{\text{the decrease of soil water reserve on domain } \left[Z_1 = 0, Z_2 = Z^* + \Delta Z\right]}{\text{the decrease of water table's level}} \quad (1)$$

For an infinitesimal decrease of water table $\Delta Z$, can be defined ***the local soil storage coefficient*** corresponding to depth $Z^*$, $\mu^* = \mu\left(Z^*\right)$, by means of the next limit (Eq. 2):

$$\mu^* = \lim_{\Delta Z \to 0} \frac{\Delta W_{1\_2}}{\Delta Z} \quad (2)$$

Between the two storage coefficients, $\overline{\mu}$ and $\mu^*$, the next integral equation can be defined (Eq. 3):

$$\overline{\mu} = \frac{1}{\Delta Z} \int_{Z^*}^{Z^*+\Delta Z} \mu^*\left(\varsigma\right) \mathrm{d}\varsigma \quad (3)$$

In this paper the authors proposed a procedure to determine the analytical equations (on basis of distribution law for piezometric height (suction) and

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parametric models for the constitutive equation suction-humidity) for a direct and highly accurate assessment of storage coefficients, $\overline{\mu}$ and $\mu$*.

## 2. The mathematical model

Currently, the constitutive equation of suction-humidity for a soil horizon, considered to be a porous, homogenous and unsaturated medium, has the form (4):

$$S_w = S_w(\psi) \text{ or } \theta_w = \theta_w(\psi) \qquad (4)$$

where:
$S_w$ = the saturation degree (saturation);
$\psi$ = the piezometric height (for $\psi \le 0, |\psi|$ represents the vacuum-metrical height or the suction), [L].

In *usual parametrical models*, the constitutive equation (4) can be expressed with the notion of *effective saturation degree*, $S_w^e$, defined as follows (Diersch, 2002) (Eq. 5):

$$S_w^e = \frac{S_w - S_w^r}{S_w^s - S_w^r} = \frac{\theta_w - \theta_w^r}{\theta_w^s - \theta_w^r}, \qquad (5)$$

where:
$S_w^s$ is the degree of maximal saturation ($S_w^s = 1$); $S_w^r$ - the degree of residual saturation;
$\theta_w$ - the volumetric humidity ($\theta_w = eS_w$, where $e$ is the voids factor); $\theta_w^s$ is the maximal humidity ($\theta_w^s = e$) and $\theta_w^r$ - the residual humidity.

The way of which the effective saturation degree $S_w^e$ depends on water's piezometric height within soil's pores, $\psi$, is given by means of empirical equations, depending on the type of parametrical.

For example:

a) for the *van Genuchten-Mualem* model,

$$S_w^e(\psi) = \begin{cases} \left[1 + \left(A|\psi - \psi_a|\right)^n\right]^{-m}, & \text{for } \psi < \psi_a \\ 1, & \text{for } \psi \ge \psi_a \end{cases} \qquad (6a)$$

where *m and n*, are exponents and *A*, $\left[L^{-n}\right]$, the model's coefficient, and $\psi_a$, [L] – the piezometric height of air within soil;

b) for the exponential model (proposed by authors),

$$S_w^e(\psi) = \begin{cases} Ee^{\alpha(\psi - \psi_a)}, & \text{for } \psi < \psi_a \\ 1, & \text{for } \psi \ge \psi_a \end{cases}, \qquad (6b)$$

where α, [L$^{-1}$], is the sorptive number, and *E*, [-], the model's coefficient (in the classic exponential model, *E* =1).

If the humidity $\theta_w$, will be taken from Eq. (5), the next expression for the constitutive will result (Eq. 7):

$$\theta_w(\psi) = \begin{cases} \theta_w^r + S_w^e(\psi)\left(\theta_w^s - \theta_w^r\right), & \text{for } \psi < \psi_a \\ \theta_w^s, & \text{for } \psi \ge \psi_a \end{cases} \qquad (7)$$

When water table is at depth $Z^*$ under soil's level, the piezometric height's distribution, in conditions of hydrodynamic balance (assumption 2º), is given by the law (8):

$$\psi = Z - Z^* + \psi_a \qquad (8)$$

therefore Eqs. (9) will result.

$$\psi - \psi_a = \begin{cases} Z - Z^* \le 0, & \text{for } Z \in \left[0, Z^*\right] \\ Z - Z^* \ge 0, & \text{for } Z \ge Z^* \end{cases}$$

$$|\psi - \psi_a| = \begin{cases} Z^* - Z, & \text{for } Z \in \left[0, Z^*\right] \\ Z - Z^*, & \text{for } Z \ge Z^* \end{cases} \qquad (9)$$

If Eqs. ((6) and (9) are considered in Eq. (7), the *humidity distribution laws upon soil's profile* will result (Eqs. 10):

a) for the *van Genuchten-Mualem* model (Eq. 10a):

$$\theta_w(Z) = \begin{cases} \theta_w^r + \dfrac{\theta_w^s - \theta_w^r}{\left[1 + A^n\left(Z^* - Z\right)^n\right]^m}, & \text{for } Z \in \left[0, Z^*\right] \\ \theta_w^s, & \text{for } Z \ge Z^* \end{cases} \qquad (10a)$$

b) for the *exponential* model (Eq. 10b):

$$\theta_w(Z) = \begin{cases} \theta_w^r + \left(\theta_w^s - \theta_w^r\right)Ee^{\alpha\left(Z - Z^*\right)}, & \text{for } Z \in \left[0, Z^*\right] \\ \theta_w^s, & \text{for } Z \ge Z^* \end{cases} \qquad (10b)$$

The water reserve stored upon soil's profile between depths $Z_1$ and $Z_2$, with $0 \le Z_1 < Z_2$, can be assessed with Eq. (11):

$$W_{1\_2} = \int_{Z_1}^{Z_2} \theta_w(Z)\mathrm{d}Z . \qquad (11)$$

In the initial situation (when water table is at depth $Z^*$), the humidity distribution upon soil's profile on depth $\left[Z_1 = 0, Z_2 = Z^* + \Delta Z\right]$ is given by Eqs. (10). Hence, the stored water reserve, given by Eq. (11), is expressed by Eqs.(12):

a) for the *van Genuchten-Mualem* model (Eq. 12a)

$$W^0_{1\_2} = \theta^r_w Z^* + \theta^s_w \Delta Z + \left(\theta^s_w - \theta^r_w\right) \int_0^{Z^*} \left[1 + \left(A\left(Z^* - Z\right)\right)^n\right]^{-m} \mathrm{d}Z \tag{12a}$$

b) for the *exponential* model (Eq. 12b)

$$W^0_{1\_2} = \theta^r_w Z^* + \theta^s_w \Delta Z + \left(\theta^s_w - \theta^r_w\right) E \int_0^{Z^*} e^{\alpha\left(Z - Z^*\right)} \mathrm{d}Z = \\ = \theta^r_w Z^* + \theta^s_w \Delta Z + \frac{E}{\alpha}\left(\theta^s_w - \theta^r_w\right)\left(1 - e^{-\alpha Z^*}\right). \tag{12b}$$

In the final situation (when water table is at depth $Z^* + \Delta Z$), it results from Eq. (8):

$$\psi - \psi_a = Z - \left(Z^* + \Delta Z\right) \le 0, \text{ for } Z \in \left[0, Z^* + \Delta Z\right] \tag{13}$$

Therefore, humidity distribution upon soil's profile, on the depth $\left[Z_1 = 0, Z_2 = Z^* + \Delta Z\right]$ is expressed by Eqs. (14):

a) for the *van Genuchten-Mualem* model (Eq. 14a):

$$\theta_w(Z) = \theta^r_w + \frac{\theta^s_w - \theta^r_w}{\left[1 + \left(A\left(Z^* + \Delta Z - Z\right)\right)^n\right]^m}, \text{ for } Z \in \left[0, Z^* + \Delta Z\right] \tag{14a}$$

b) for the *exponential* model (Eq. 14b):

$$\theta_w(Z) = \theta^r_w + \left(\theta^s_w - \theta^r_w\right) E e^{\alpha\left(Z - Z^* - \Delta Z\right)}, \text{ for } Z \in \left[0, Z^* + \Delta Z\right] \tag{14b}$$

Therefore, in the final situation, from Eq. (11) it results the following stored water reserve on the depth $\left[Z^*, Z^* + \Delta Z\right]$ (Eqs. 15):

a) for the *van Genuchten-Mualem* model (Eq. 15a):

$$W^F_{1\_2} = \theta^r_w\left(Z^* + \Delta Z\right) + \left(\theta^s_w - \theta^r_w\right) \int_0^{Z^* + \Delta Z} \left[1 + \left(A\left(Z^* + \Delta Z - Z\right)\right)^n\right]^{-m} \mathrm{d}Z \tag{15a}$$

b) for the *exponential* model (Eq. 15b):

$$W^F_{1\_2} = \theta^r_w\left(Z^* + \Delta Z\right) + \frac{E}{\alpha}\left(\theta^s_w - \theta^r_w\right)\left[1 - e^{-\alpha\left(Z^* + \Delta Z\right)}\right] \tag{15b}$$

Taking into account Eqs. (12) and (15), the basic equation (1) for the ***average storage coefficient*** becomes (16):

$$\overline{\mu} = \frac{W^0_{1\_2} - W^F_{1\_2}}{\Delta Z} \tag{16}$$

In case of the *van Genuchten-Mualem* model, generally, the integrals within 12th and 15th equations can be determined only by means of numerical methods; thus the average storage coefficient $\overline{\mu}$ will be determined too by means of numerical methods. However, in the case of the exponential model, when using for $W^0_{1\_2}$ and $W^F_{1\_2}$ and the analytical equations (12-b) and (15-b), the following equation for the $\overline{\mu}$ coefficient results (Eq. 17):

$$\overline{\mu} = \left(\theta^s_w - \theta^r_w\right)\left[1 - \frac{E}{\alpha\,\Delta Z} e^{-\alpha Z^*}\left(1 - e^{-\alpha\,\Delta Z}\right)\right]. \tag{17}$$

In a similar way, the basic equation (2) for the *local storage coefficient* $\mu^*$, becomes (18):

$$\mu^* = \lim_{\Delta Z \to 0} \frac{W^0_{1\_2} - W^F_{1\_2}}{\Delta Z}. \tag{18}$$

If equations (12) and (15) are incorporated within equation (18), and if the rules for differential and integral calculation are applied, algebraic equations for $\mu^*$ coefficient will result (Eqs. 19):

a) for the *van Genuchten-Mualem* model (Eq. 19a):

$$\mu^* = \left(\theta^s_w - \theta^r_w\right)\left\{1 - \left[1 + \left(A \cdot Z^*\right)^n\right]^{-m}\right\} \tag{19a}$$

b) for the *exponential* model (Eq.19b):

$$\mu^* = \left(\theta^s_w - \theta^r_w\right)\left(1 - E e^{-\alpha Z^*}\right). \tag{19b}$$

Eq. (19-b) can also be obtained directly, applying the *L'Hospital* rules to the limit given below (Eq. 19c):

$$\mu^* = \lim_{\Delta Z \to 0} \left(\theta^s_w - \theta^r_w\right)\left[1 - \frac{E}{\alpha\,\Delta Z} e^{-\alpha Z^*}\left(1 - e^{-\alpha\,\Delta Z}\right)\right] \tag{19c}$$

In case of the Genuchten-Mualem model, generally, for the $m$ and $n$ exponents, from the 6th law, it can be seen that: $m \notin \mathbf{Z}$ and $(m+1)/n \notin \mathbf{Z}$, where $\mathbf{Z}$ is the integer set.

Therefore the binomial integrals within Eqs. 12, 15 and 3 do not satisfy the precise analytical integration conditions deducted by Tchebyshev; hence, all these integrals are to be solved numerically.

In the particular case when the trapeze formula and, respectively, the rectangle formula are applied, formal approximate analytical equations will result from Eq. (3) for the storage coefficient $\overline{\mu}$ (Eqs. 20a and b):

$$\overline{\mu} \cong \frac{1}{2}\left[\mu^*(Z_1)+\mu^*(Z_2)\right]= =\left(\theta_w^s-\theta_w^r\right)\left\{2-\left[1+\left(AZ^*\right)^n\right]^{-m}-\left[1+A^n\left(Z^*+\Delta Z\right)^n\right]^{-m}\right\} \quad (20a)$$

and

$$\overline{\mu} \cong \mu^*\left(Z^*+\frac{\Delta Z}{2}\right)=\left(\theta_w^s-\theta_w^r\right)\left\{1-\left[1+A^n\left(Z^*+\frac{\Delta Z}{2}\right)^n\right]^{-m}\right\} \quad (20b)$$

## 3. The numerical application

In order to test the proposed mathematical model the next basic data were considered:

- The nature of soil horizon: clay, featuring the next physical and hydrical parameters: porosity, $n$ = 0.33643 (or voids factor, $e$ = 0.507); volumetric humidity, [expressed in $m^3$ water / $m^3$ solid phase], residual, $\theta_w^r$=0.18252; maximal volumetric humidity, $\theta_w^s = e$ = 0.507; air piezometric height, $\psi_a$=0.
- The constitutive equation (4): given (Table 1 – columns 2 and 3) by the next pairs of values:

$$\left(\psi,\theta_w\right)_i=\left(\psi_i,\theta_{wi}\right), \text{ with } i=1,2,\ldots, N_{date}=10. \quad (21)$$

At the initial moment the water table level is at the depth $Z^*$=50 cm, and at final moment the level descends with an amount $\Delta Z$=70 cm.

The next calculation stages have been carried:

- By means of a proprietary software, **functia_genuchten.m**, using the input data (21), the $A$ coefficient can be computed and the power exponents $n$, $m$ for law (6a) result:

$$A = 3.163067198535394\text{e-}4, \quad n = 0.538301890103307, \quad m = 0.999999999999965 \quad (22)$$

- By means of a proprietary software, **functia_exponent.m**, with input data (21), he $E$ coefficient and the $\alpha$ exponent for law (6b) can be computed:

$$E=0.964379348962526,\ \alpha=0.001128727262118. \quad (23)$$

- Incorporating in Eq. (6a) the values of $A$, $n$ and $m$ computed by (22) and the values from (21) for $\psi_i$ – in equation (7), the values for humidity $\theta_w$ result approximated with Genuchten-Mualem model (Table 1 – column 4):

$$\theta_{wi}^{G\_M}, \text{ with } i=1,2,\ldots, N_{date}=10. \quad (24)$$

Incorporating in Eq. (6b) the values of $E$ and $\alpha$ given by (23) and the values from (21) for $\psi_i$ – in Eq. (7), the values of humidity $\theta_w$ are obtained, approximated with the exponential model (Table 1 – column 5) (Eq. 25):

$$\theta_{wi}^{Exp}, \text{ with } i=1,2,\ldots, N_{date}=10. \quad (25)$$

The relative errors corresponding to the approximation with the two models - Genuchten-Mualem, $\varepsilon_i^{G\_M}$, and exponential, $\varepsilon_i^{Exp}$ - evaluated, respectively, with equations (26):

$$\varepsilon_i^{G\_M}=100\frac{\theta_{wi}^{G\_M}-\theta_{wi}}{\theta_{wi}}, \ i=1,2,\ldots,N_{date}=10, \quad [\%]$$

and (26)

$$\varepsilon_i^{Exp}=100\frac{\theta_{wi}^{Exp}-\theta_{wi}}{\theta_{wi}}, \text{ with } i=1,2,\ldots,N_{date}=10, \quad [\%]$$

centralized in Table 1- columns 6 and 7.

- After assessment of data within Table 1 - columns 6 and 7, it can be seen that the maximal relative error, in the module (absolute value), introduced by the Genuchten-Mualem model, is much inferior to the one corresponding to the exponential model,
- $$\left|\varepsilon_9^{G\_M}\right| = 0.2019 \ll \left|\varepsilon_{10}^{Exp}\right| = 1.2503.$$

For this reason the results obtained with the Genuchten-Mualem model has been considered as reference results.

- Also the depths are computed (Eq. 27):

$$Z_1=Z^*=50 \text{ cm},\ Z_2=Z^*+\Delta Z=120 \text{ cm},\ Z_{med}=\left(Z_1+Z_2\right)/2=85 \text{ cm}. \quad (27)$$

**Table 1**. Experimental and theoretical values (the Genuchten-Mualem and exponential models) for the suction-humidity curve

| ***i*** | ***1*** | ***2*** | ***3*** | ***4*** | ***5*** | ***6*** | ***7*** | ***8*** | ***9*** | ***10*** |
|---|---|---|---|---|---|---|---|---|---|---|
| $\psi_i$,[cm] | -120 | -100 | -80 | -70 | -50 | -40 | -30 | -20 | -10 | -2 |
| $\theta_{w\,i}$, [-] | 0.4590 | 0.4630 | 0.4680 | 0.4700 | 0.4760 | 0.4790 | 0.4830 | 0.4870 | 0.4920 | 0.5010 |
| $\theta_{wi}^{G\_M}$, [-] | 0.4594 | 0.4633 | 0.4676 | 0.4700 | 0.4756 | 0.4788 | 0.4825 | 0.4870 | 0.4930 | 0.5010 |
| $\theta_{wi}^{Exp}$, [-] | 0.4558 | 0.4620 | 0.4684 | 0.4717 | 0.4783 | 0.4816 | 0.4850 | 0.4885 | 0.4919 | 0.4947 |
| $\varepsilon_i^{G\_M}$ [%] | 0.0889 | 0.0556 | -0.0839 | 0.0060 | -0.0925 | -0.0399 | -0.0934 | 0.0099 | **0.2019** | -0.0081 |
| $\varepsilon_i^{Exp}$,[%] | -0.6966 | -0.2069 | 0.0905 | 0.3551 | 0.4770 | 0.5485 | 0.4189 | 0.2992 | -0.0143 | **-1.2503** |

**Table 2**. Values for the average and local storage coefficients

| ***Nr. Crt*** | ***Parameter*** | ***The Genutchten-Mualem model*** | | | ***The exponential model*** | | |
|---|---|---|---|---|---|---|---|
| | | ***Equa-tion*** | ***Value*** | ***Error [%]*** | ***Equa-tion*** | ***Value*** | ***Error [%]*** |
| 1 | $W_{1\_2}^{0}$ | (12a) | 59.7913380636225 | 0.0000 | (12b) | 59.8287760899752 | 0.0626 |
| 2 | $W_{1\_2}^{F}$ | (15a) | 56.977067476729 | 0.0000 | (15b) | 57.0209851251355 | 0.0771 |
| 3 | $\overline{\mu}$ | (16), (3) | 0.04020386552705 | 0.0000 | (16), (17) | 0.0401112994977 | -0.2302 |
| 4 | $\mu^*(Z^*)$ | (19a) | 0.03144031900876 | 0.0000 | (19b) | 0.02872926395388 | -8.6229 |
| 5 | $\mu^*(Z^*+\Delta Z)$ | (19a) | 0.0475919176681 | 0.0000 | (19b) | 0.05119749328637 | -15.5629 |
| 6 | $\mu^*\left(Z^*+\dfrac{\Delta Z}{2}\right)$ | (19a) | 0.04053627817868 | 0.0000 | (19b) | 0.04018525416554 | -0.8660 |
| 7 | $\overline{\mu}$ | (20a) | 0.03951611833843 | -1.7106 | - | | |
| 8 | $\overline{\mu}$ | (20b) | 0.04053627817868 | 0.8268 | - | | |

By analyzing data from Table 2, the next issues could be highlighted:

1º The assessment of the average storage coefficient $\overline{\mu}$ with Eq. (17) leads to relative errors, in module, inferior to 0.25%;

2º The assessment of the local storage coefficient $\mu^*$, with equation 19-b), in extreme positions of water table ($Z^*$ and $Z^*+\Delta Z$), leads to non-acceptable relative errors;

3º The assessment of the $\overline{\mu}$ storage coefficient with the approximate Eqs. (20) may lead to relative errors, in module, exceeding 1%.

## 4. Original contributions and conclusions

On basis of obtained results the next contributions and conclusions can be highlighted:

1. A rigorous definition of the average and local storage coefficients.

2. For the suction-humidity soil constitutive equation has been presented and implemented an original enhanced exponential model.

3. The authors are recommending the assessment of the average storage coefficient $\overline{\mu}$ by means of Eq. (17) deducted on basis of an enhanced exponential model of suction-humidity constitutive equation.

4. The authors are recommending the assessment of the local storage coefficient $\mu^*$ by means of the Eq. (19a) – deducted on basis of the Genuchten - Mualem model of suction-humidity constitutive equation.

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